Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

CONJUGATE DIRECTIONS ON A HYPERSURFACE IN A SPACE OF FOUR DIMENSIONS AND SOME ALLIED CURVES.

BY C. L. E. MOORE.

Introduction.—On a surface in a space of three dimensions the line joining two infinitely near points (x_1, x_2, x_3, x_4), $(x_1 + dx_1, x_2 + dx_2 \cdots)$, or as we shall say for brevity the points x and $x + dx$, and the line of intersection of the planes tangent to the surface at these two points define what are called conjugate directions. The correspondence between these lines is projective and the coincident or self-corresponding elements define the principal or asymptotic directions. This is one of the most important topics in differential geometry.

The idea of conjugate directions has been somewhat generalized by Segre.* He studied the correspondence between the planes determined by the fixed point x and the two neighboring points $x + dx$, $x + 2dx + d^2x$ (dx and d^2x are variable) and the point of intersection of the planes tangent to the surface at these points.

It is the object of the present paper to study the two corresponding problems for hypersurfaces in space of four dimensions. In each case we have a line-plane correspondence, which possesses properties similar to those for space of three dimensions. The correspondence between the planes determined by the fixed point x and the points $x + dx$, $x+2dx+d^2x$ and the line of intersection of the plane spaces tangent to the hypersurface at these three points forms the extension of Segre's problem. This correspondence is (1, 1). The correspondence between the plane S_3 determined by four infinitely near points and the point of intersection of the tangent spaces to the hypersurface at these points is (∞, ∞) and therefore is not discussed. Here four infinitely near points uniquely determine a single point of intersection of the tangent spaces but the same S_3 is determined by any four infinitely near points lying on the surface of intersection of the hypersurface with this S_3.

1. Conjugate Directions.—The tangent lines at a point x to a hypersurface in a space of four dimensions S_4 will generate a tangent hyperplane π_3. Two successive tangent hyperplanes will intersect in a plane which is tangent to the hypersurface at x, that is, the plane will contain a pencil of tangent lines passing through x. The common plane and the line joining the points of contact of two successive tangent hyperplanes are said to be *conjugate*. If we consider the hypersurface as the envelope of its tangent hyperplanes, the tangent π_3 at the point x and the tangent π_3 at the infinitely near point $x + dx$ can be taken as the hyperplanes ξ and $\xi + \delta\xi$ where the equation of the hyperplane is $\Sigma\xi x = 0$. If we consider a direction as defined by two infinitely near elements, whether points or hyperplanes, the two hyperplanes ξ and $\xi + \delta\xi$ will define a direction among the ∞^3 hyperplanes tangent to the hypersurface V_3. Then we will say that the directions x, $x + dx$ and ξ, $\xi + \delta\xi$ are conjugate. This agrees with the general conception of conjugate directions on a surface in an ordinary space S_3.

---

* Complementi alla theoria delle tangenti coniugati di una superficie. Rendiconti dei Lincei, vol. XVII, page 405.

We saw that the common plane to two consecutive hyperplanes contains a pencil of lines (directions) through the point O. There we can look upon the line joining the points of contact as conjugate to this whole pencil of directions. This correspondence between the tangent lines and the tangent planes is a $(1, 1)$ correspondence. In fact, it is a polar reciprocation as can be shown as follows.

Let the equations of the hypersurface V_3 be

$$(F) \qquad x_i = x_i(u_1, u_2, u_3) \qquad (i = 1, 2, 3, 4, 5).$$

Now if we consider the u's as functions of a single variable t, x_i will describe a curve on V_3 passing through the point O whose coordinates are $(x_1, x_2, x_3, x_4, x_5)$ say. The tangent line to this curve will join x to $x + dx$. On the other hand if we consider V_3 as the envelope of its tangent hyperplanes then its equations are

$$(E) \qquad \xi_i = \xi_i(u_1, u_2, u_3) \qquad (i = 1, 2, 3, 4, 5).$$

In these equations (F) and (E) if we give any definite set of values to the u's the equation (F) will give the point of contact of the hyperplane (E). If the u's in (E) are the same functions of t as above, the hyperplanes (E) will be tangent to V_3 in the points of the above curve. Two successive tangent hyperplanes will intersect in a tangent plane and the correspondence sought is that between these planes and the tangent lines to the curve at the point of contact of the plane.

Consider the two hyperplanes obtained by giving to the parameters the values u and $u + \delta u$. These same values of the parameters put in equations (F) will give the points of contact of the two hyperplanes. Then on V_3 the direction determined by δu_i is the direction of the line joining the points of contact of the two hyperplanes. The equations of the two hyperplanes are

$$(1) \qquad (\xi, x) = 0, \quad (\xi + \delta\xi, x) = 0,$$

where

$$(\xi, x) = \Sigma \xi_i x_i = \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 + \xi_4 x_4 + \xi_5 x_5.$$

The second of equations (1) can be replaced, in consequence of the first, by

$$(\delta\xi, x) = 0.$$

The condition that this hyperplane should contain the point $x + dx$, since it already contains the point x is

$$(\delta\xi, dx) = 0.$$

The directions du are then the directions conjugate to the directions δu. All the directions du which are conjugate to δu form the plane of intersection of the two hyperplanes ξ and $\xi + \delta\xi$. The above relation may be put in the form

$$\left[\left(\frac{\partial\xi}{\partial u}, \delta u\right), \left(\frac{\partial x}{\partial u}, du\right)\right] = 0.$$

or

$$\Sigma\left(\frac{\partial\xi}{\partial u_i}, \frac{\partial x}{\partial u_j}\right) \delta u_i du_j = 0. \tag{2}$$

The tangent hyperplane is determined by the points

$$(\xi, x) = 0, \quad \left(\xi, \frac{\partial x}{\partial u_1}\right) = 0, \quad \left(\xi, \frac{\partial x}{\partial u_2}\right) = 0, \quad \left(\xi, \frac{\partial x}{\partial u_3}\right) = 0 \tag{3}$$

(here the points are represented as the envelope of the hyperplanes which pass through them). Differentiating equations (3) except the first and remembering that both the ξ's and the x's are functions of the u's, we have

$$\left(\frac{\partial\xi}{\partial u_i}, \frac{\partial x}{\partial u_j}\right) + \left(\xi, \frac{\partial^2 x}{\partial u_i \partial u_j}\right) = 0 \qquad (i, j = 1, 2, 3).$$

Substituting the values of $\left(\frac{\partial\xi}{\partial u_i}, \frac{\partial x}{\partial u_j}\right)$ from these equations in (2), we have

$$\sum_{i,j}\left(\xi, \frac{\partial^2 x}{\partial u_i du_j}\right) \delta u_i du_j = 0,$$

or in still simpler form

$$\left(\xi, \Sigma \frac{\partial^2 x}{\partial u_i \partial u_j} \delta u_i du_j\right) = 0. \tag{4}$$

Eliminating ξ between (3) and (4) we have

$$(5)\qquad \begin{vmatrix} x_1 & \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \frac{\partial x_1}{\partial u_3} & \Sigma \frac{\partial^2 x_1}{\partial u_i \partial u_j} \delta u_i du_j \\ x_2 & \frac{\partial x_2}{\partial u_1} & \cdot & \cdot & \cdots \\ x_3 & \frac{\partial x_3}{\partial u_1} & \cdot & \cdot & \cdots \\ x_4 & \frac{\partial x_4}{\partial u_1} & \cdot & \cdot & \cdots \\ x_5 & \frac{\partial x_5}{\partial u_1} & \cdot & \cdot & \cdots \end{vmatrix} = 0.$$

In this equation δu and du appear symmetrically and therefore the correspondence between the planes defined by δu and the lines defined by du is symmetric.

When $\delta u = du$, we have

$$(5') \qquad \left| x_1 \frac{\partial x_1}{\partial u_1} \frac{\partial x_1}{\partial u_2} \frac{\partial x_1}{\partial u_3'} \Sigma \frac{\partial^2 x_1}{\partial u_i \partial u_j} du_i du_j \right| = 0.$$

This new equation defines a quadric cone of directions in the tangent π_3. The planes corresponding to the lines of this cone will contain their corresponding line and are tangent planes to the cone. The plane corresponding to any line (direction) through the vertex of the cone is the polar reciprocal of the line with respect to the cone. These properties can be most easily seen by cutting π_3 by an S_3 which does not pass through x. Then in the plane of intersection δu_i may be considered as the coordinates of a line and du_i as the coordinates of a point. Equation (5) defines a polar reciprocation from which the above statements follow if we replace lines of (5′) by points of a conic and planes tangent to (5′) by lines tangent to the conic *ecc.*

The tangents to the directions defined by the cone (5′) will envelope curves on V_3 which have the following properties:

(1) *The osculating planes to such a curve are tangent*[*] *to* V_3.

(2) *The tangents to these curves have three point contact with the hypersurface.*

In order to establish these theorems we consider three infinitely near points of the hypersurface which we shall represent in hyperplanar coordinates.[†] The expression

---

[*] A plane is said to be tangent to a hypersurface if it lies in the tangent hyperplane and passes through the point of contact.

[†] See Segre: Su una classa di Superficie ecc. Atti di Torino, 1907.

$$f = (\xi, x)$$

will be taken to represent the point x. The quantities ξ are hyperplanar coordinates and are not functions of u_1, u_2, u_3. With this notation the three points infinitely near are

$$f = (\xi, x) = 0, \tag{6}$$

$$f_1 du_1 + f_2 du_2 + f_3 du_3 = 0, \tag{7}$$

$$\sum_{ik}^{1,2,3} f_{ik} du_i du_k + \sum_{1}^{3} f_i d^2 u_i = 0 \tag{8}$$

where

$$f_i = \frac{\partial f}{\partial u_i}, \quad f_{ij} = \frac{\partial^2 f}{\partial u_i \partial u_j}.$$

The tangent π_3 is determined by the points

$$f = 0, \quad f_1 = 0, \quad f_2 = 0, \quad f_3 = 0.$$

Points (6) and (7) already lie in this π_3 and we see that (8) is a point on the line joining

$$\Sigma f_{ij} du_i du_j = 0, \quad \Sigma f_i d^2 u_i = 0,$$

the second of which we see always lies in the tangent π_3. Hence the plane determined by (6), (7), (8) will lie in the tangent π_3 if the five points

$$f = 0, \quad f_1 = 0, \quad f_2 = 0, \quad f_3 = 0, \quad \Sigma f_{ik} du_i du_k = 0 \tag{T}$$

lie in this π_3. This condition is exactly equation (5′). Hence

Through each point of V_3 pass ∞^1 curves for each of which the osculating plane is tangent to V_3.

To prove the second property, that is, that tangents to these curves have three point contact with V_3 we will investigate the condition in order that the three points (6), (7), (8) lie in the same line. It will be necessary to show that if these three points lie in a line the five points (T) will lie in a π_3 and conversely if the five points (T) lie in a π_3 (6), (7), (8) are collinear. The first is established as follows. (8) represents a point on the line joining the two points $\Sigma f_{ik} du_i du_k = 0$, $\Sigma f_i d^2 u_i = 0$. This second point is some point of the plane determined by $f_1 = f_2 = f_3 = 0$. The point (7) is also a point of this plane. Then the line joining (7) and (8) must lie in the S_3 determined by the four points $\Sigma f_{ik} du_i du_k = 0$, $f_1 = 0$, $f_2 = 0$, $f_3 = 0$, and if the points (6), (7), (8) are collinear then $f = 0$ must also lie in this same S_3. Equation (5′) is the condition that these five points should lie in the same S_3. Conversely if the above five points lie in an S_3 then $d^2 u_i$ can be determined so that the point (8) will lie on the line joining (6) and (7). For if d^2u_i are allowed to vary (8) will generate a plane in π_3 and the values of d^2u_i which will give the point where the line joining (6) and (7) cuts this plane will be the values desired. Hence:

The curves tangent to the elements of the cone (5′) *are such that their tangents have three point contact with* V_3.

If we trace on V_3 any curve c the planes conjugate to the tangent lines of the curve will envelope a developable* surface.

This is evident from the fact that the tangent π_3's to V_3 in points of c, admit as envelope a system of ∞^1 planes. The envelope of this system of planes is a system of ∞^1 lines which touch a curve.

2. **Asymptotic Directions.**—The ∞^1 directions defined by (5′) can be looked upon as directions along which the tangent hyperplane π_3 has higher order contact with V_3. In fact along these directions π_3 contains points of V_3 infinitely near to the second order, since it contains the osculating planes to the curves in this direction. Let us now examine if there are directions along which π_3 will have contact of third order, that is the osculating S_3 of curves in this direction will coincide with π_3. The osculating S_3 of a curve on V_3 passing through a point O given by (6) is defined by (6), the points (7), (8) and (9)

$$\Sigma f_{ijk}du_idu_jdu_k + 3\Sigma f_{ik}du_id^2u_k + \Sigma f_id^3u_i = 0.$$

If the first three points lie in π_3, it is necessary that

$$\Sigma f_{ik}du_idu_k = 0 \tag{10}$$

also lies in π_3. Writing this in the form

$$(f_{11}du_1 + f_{12}du_2 + f_{13}du_3)du_i + (f_{12}du_1 + f_{22}du_2 + f_{23}du_3)du_2 \\ + (f_{13}du_1 + f_{23}du_2 + f_{33}du_3)du_3 = 0,$$

and writing the second term of (9) in the form

$$(f_{11}du_1 + f_{12}du_2 + f_{13}du_3)d^2u_1 + (f_{12}du_1 + f_{22}du_2 + f_{23}du_3)d^2u_2 \\ + (f_{13}du_1 + f_{23}du_2 + f_{33}du_3)d^2u_3 = 0, \tag{10′}$$

we see that if (10) lies in π_3 then d^2u_i can be so chosen that this latter point also will lie in it. Since $\Sigma f_id^3u_i = 0$ lies in π_3, in order for the four points to lie in π_3, it remains only for

$$\Sigma f_{ijk}du_idu_jdu_k = 0$$

to lie in it. Then we have the condition (5′) and

---

* A developable surface here used is a surface which has the same tangent plane at every point of a line.

$$(11) \qquad \begin{vmatrix} x_1 & \dfrac{\partial x_1}{\partial u_1} & \dfrac{\partial x_1}{\partial u_2} & \dfrac{\partial x_1}{\partial u_3} & \Sigma \dfrac{\partial^3 x_1}{\partial u_1 \partial u_2 \partial u_3} du_1 du_2 du_3 \\ x_2 & \dfrac{\partial x_2}{\partial u_1} & \cdot & \cdot & \cdot \\ x_3 & \dfrac{\partial x_3}{\partial u_1} & \cdot & \cdot & \cdot \\ x_4 & \dfrac{\partial x_4}{\partial u_1} & \cdot & \cdot & \cdot \\ x_5 & \dfrac{\partial x_5}{\partial u_1} & \cdot & \cdot & \cdot \end{vmatrix} = 0.$$

Then the directions along which the osculating S_3 to the curve coincides with π_3 are defined by the values of $du_1 \colon du_2 \colon du_3$ which satisfy (5′) and (11). Hence

Through a point O on V_3 *pass six directions** *along which curves can be drawn whose osculating* S_3 *coincides with the tangent* π_3 *at O.*

We saw that equation (5′) expressed the condition that the three infinitely near points (x), (dx), (d^2x) should be on the same line, or that the points (6), (7), (8) which determine the plane of these three points should be on the same line. Equation (11) expresses that the point (9) also lies on the same line. For (9) is some point in the plane determined by the three points

$$\Sigma f_{ijk} du_i du_j du_k = 0, \quad \Sigma f_{ij} du_i d^2u_j = 0, \quad \Sigma f_i d^3u_i = 0.$$

The second and third points lie on the line determined by (6), (7), (8) for particular values of d^2u_i and d^3u_i. Then if (9) is to lie on this line, it is only necessary to have $\Sigma f_{ijk} du_i du_j du_k = 0$ on the line which condition is expressed by equation (11). Hence

Through each point of a hypersurface pass six directions such that the tangent line to curves in this direction has four point contact with the hypersurface.

---

* This can also be seen by expanding the coordinates about the point O. The point $x+dx$ corresponds to the value $u+du$ of the parameters; then

$$x + dx = x + \Sigma \frac{\partial x}{\partial u} du + \frac{1}{2} \Sigma \frac{\partial^2 x}{\partial u_i \partial u_j} du_i du_j + \frac{1}{6} \Sigma \frac{\partial^3 x}{\partial u_i \partial u_j \partial u_k} du_i du\, du_k + \cdots.$$

The tangent hyper-plane contains $x+dx$ to first order infinitesimals and the cone defined by (5′) to second order and if the tangent hyperplane contains

$$\Sigma \frac{\partial^3 x}{\partial u_i \partial u_j \partial u_k} du_i du_j du_k$$

it will contain $x+dx$ to third order infinitesimals.

These curves form the real generalization of asymptotic curves on a surface in ordinary space.

If we expand equation (5′) in terms of du_i we have

$$(5'')\quad P du_1^2 + Q du_2^2 + R du_3^2 + 2L du_1 du_2 + 2M du_1 du_3 + 2N du_2 du_3 = 0$$

as the equation of the cone of principal directions where

$$P = \left| x_1 \frac{\partial x_1}{\partial u_1} \frac{\partial x_1}{\partial u_2} \frac{\partial x_1}{\partial u_3} \frac{\partial^2 x_1}{\partial u_1^2} \right|, \quad Q = \left| x_1 \frac{\partial x_1}{\partial u_1} \frac{\partial x_1}{\partial u_2} \frac{\partial x_1}{\partial u_3} \frac{\partial^2 x_1}{\partial u_2^2} \right|,$$

$$R = \left| x_1 \frac{\partial x_1}{\partial u_1} \frac{\partial x_1}{\partial u_2} \frac{\partial x_1}{\partial u_3} \frac{\partial^2 x_1}{\partial u_3^2} \right|, \quad L = \left| x_1 \frac{\partial x_1}{\partial u_1} \frac{\partial x_1}{\partial u_2} \frac{\partial x_1}{\partial u_3} \frac{\partial^2 x_1}{\partial u_1 \partial u_2} \right|,$$

$$M = \left| x_1 \frac{\partial x_1}{\partial u_1} \frac{\partial x_1}{\partial u_2} \frac{\partial x_1}{\partial u_3} \frac{\partial^2 x}{\partial u_1 \partial u_3} \right|, \quad N = \left| x_1 \frac{\partial x_1}{\partial u_1} \frac{\partial x_1}{\partial u_2} \frac{\partial x_1}{\partial u_3} \frac{\partial^2 x_1}{\partial u_2 \partial u_3} \right|.$$

This cone will degenerate if

$$(12)\qquad \begin{vmatrix} P & L & M \\ L & Q & N \\ M & N & R \end{vmatrix} = 0.$$

Since $P, Q, R, \ldots$ are functions of the three parameters, usually on V_3 there is a surface along which the cone (5′) will factor.

3. **Special Hypersurfaces.**—Let us examine what happens to this cone of principal directions for two particular kinds of hypersurfaces:

(1) A hypersurface generated by ∞^1 planes. In this case the planes will all touch a fixed curve. For two consecutive planes will intersect in a point and the planes all touch the locus of this point.*
The equation of the hypersurface can then be written

$$(H)\qquad x_i = \lambda_i(u_1) + \lambda_i'(u_1) u_2 + \mu_i(u_1) u_3,$$

where the accent indicates differentiation with respect to u_1. From this equation it is at once evident that

$$Q = R = N = 0,$$

but that P, L, M are different from zero. Hence the cone of principal directions degenerates into two distinct planes. It is evident that one of the planes is the generator passing through the point of contact. The equations of the two planes can be written down at once. From (5″)

$$P du_1^2 + 2L du_1 du_2 + 2M du_1 du_3 = 0.$$

---

* See Segre: Preliminari di una teoria delle varieta luoghi di spazi. Rendiconti di Palermo, vol. XXX, p. 87.

The two planes then are

$$(G) \qquad du_1 = 0$$

$$(A) \qquad Pdu_1 + 2Ldu_2 + 2Mdu_3 = 0.$$

From the first we have at once the equation of the generator

$$u_1 = \text{const.}$$

The direction of the line of intersection of these two planes is

$$\frac{du_2}{du_3} = -\frac{M}{L}.$$

Substituting the values from (*H*) in the expression for *L* and *M* we have

$$L = |\lambda_1\mu_1'\lambda_1'\mu_1\lambda_1''|\,u_3,$$

$$M = |\lambda_1\lambda_1''\lambda_1'\mu_1\mu_1'|\,u_2.$$

Hence the above direction reduces to

$$\frac{u_2}{u_3} = \frac{du_2}{du_3}.$$

This defines the direction of the line in a plane $u_1 = \text{const.}$ joining the point (u_2, u_3) to the point $u_2 = 0$, $u_3 = 0$. [In the generator which passes through the point in question u_2, u_3 are the non-homogeneous coordinates of the points. The point where the generator touches the directrix (the envelope of the planes) has coordinates (0, 0).] Hence the line of intersection of the generator (*G*) with (*A*) always passes through the point where (*G*) is tangent to the directrix,

$$(C) \qquad x_i = \lambda_i(u_1).$$

If we write the equation of the tangent S_3 at the point (u_1, u_2, u_3) in the form

$$|\lambda_1 u_2\lambda_1'' + u_3\mu_1' \quad \lambda_1' u_1 x_1| = 0,$$

we see that if u_1 is held fixed and $u_2 : u_3$ is constant the tangent S_3 will be the same for the whole line thus determined. Hence a tangent S_3 to V_3 (locus of ∞^1 planes) is tangent the whole length of a line passing through the intersection of (*G*) and (*C*).*

Then the planes (*A*) corresponding to points on a line passing through the intersections of (*G*) and (*C*) form a pencil in the tangent S_3.

---

* The pencil of tangent hyperplanes passing through (*G*) are projective with the pencil of lines of contact in (*G*). This correspondence is an extension of that of Chasles for ruled surfaces in S_3. This is however only a special case of the correspondence mentioned by Segre, Preliminari di una teoria ecc. Rendiconti di Palermo, vol. XXX.

(2) If the system consists of the osculating planes of a curve

$$(C) \qquad x_i = \lambda_i(u_1),$$

then the equation of the V_3 becomes

$$x_i = \lambda_i(u_1) + u_2\lambda_i'(u_1) + u_3\lambda_i''(u_1).$$

In this case we see at once

$$Q = R = L = M = N = 0.$$

Hence the two planes of principal directions coincide. This hypersurface is a hyperdevelopable that is the envelope of ∞^1 hyperplanes. Consecutive planes of the system intersect in a line. The surface formed by the lines of intersection which we will call the singular developable might be compared to the cuspidal edge of a developable in S_3. An S_3 will cut V_3 in a developable and the edge of regression is the curve in which S_3 cuts this singular developable.

4. **Complement of Conjugate Directions.**—To the correspondence between the plane of intersection of two consecutive tangent spaces and the line joining their points of contact one obtains a complement by considering the line of intersection of three successive tangent spaces and the plane determined by the three points of tangency.*

In this way lines in the tangent space π_3 at a point O are made to correspond to the planes which pass through O. This correspondence is (1, 1). Some of the properties can be obtained immediately, e. g., to the planes passing through a line t which is tangent to V_3 in O will correspond the lines of the plane p the conjugate of t in the correspondence of section 1.

In order to obtain the equations of the correspondence, we shall consider the tangent cone drawn to V_3 from a line of π_3. Here a cone means ∞^1 planes which have a line in common. Then the osculating planes of the curve of contact of V_3 with the cone will correspond to the line which forms the vertex of the cone. We will take the point (1, 0, 0, 0, 0) as O and take the hyperplane $x_5 = 0$ as π_3. Let the equation of V_3 in homogeneous coordinates be

$$f(x_1, x_2, x_3, x_4, x_5) = f = 0.$$

The π_3 tangent at $(x_1, x_2, x_3, x_4, x_5)$ is

$$\Sigma f_i x_i = 0$$

(f_i is used for the derivative $\partial f/\partial x_i$). And if $x_5 = 0$ is to be tangent at the point (1, 0, 0, 0, 0) then

---

* See Segre: Complementi alla teoria delle tangenti coniugati di una superficie. Rendicont dei Lincei, vol. XVII, p. 405.

$$f_1 = f_2 = f_3 = f_4 = 0 \tag{13}$$

at this point. Let $(y_1, y_2, y_3, y_4, y_5)$ and $(z_1, z_2, z_3, z_4, z_5)$ denote two points in π_3. The curve of contact of the tangent cone which has the line joining these two points for vertex will be the intersection of the three hypers

$$\begin{aligned} f &= 0, \\ \Sigma y_i f_i &= 0, \\ \Sigma z_i f_i &= 0. \end{aligned} \tag{K}$$

The line of our correspondence is the line joining the points y and z and the corresponding plane is the osculating plane of (K) at O. We shall use the Plücker coordinates for the lines of π_3 $(x_5 = 0)$ which are the determinants of the matrix

$$\left\| \begin{matrix} y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{matrix} \right\| \tag{L}$$

and which we shall indicate as usual by

$$p_{ik} = y_i z_k - y_k z_i.$$

We shall use as the coordinates of the planes the determinants of the matrix of the three points O, dx/dt, d^2x/dt^2 where dx/dt and d^2x/dt^2 are determined from the equations of (K). (Here it is assumed that t is the parameter in terms of which we have expressed the coordinates of (K)). Now as the coordinates of O are $(1, 0, 0, 0, 0)$ the coordinates of the planes passing through O are the three row determinants of the matrix

$$\left\| \begin{matrix} 1 & 0 & 0 & 0 & 0 \\ \frac{dx_1}{dt} & \frac{dx_2}{dt} & \frac{dx_3}{dt} & \frac{dx_4}{dt} & \frac{dx_5}{dt} \\ \frac{d^2x_1}{dt^2} & \frac{d^2x_2}{dt^2} & \frac{d^2x_3}{dt^2} & \frac{d^2x_4}{dt^2} & \frac{d^2x_5}{dt^2} \end{matrix} \right\|.$$

Thus the coordinates of the planes are

$$q_{ik} = \frac{dx_i}{dt}\frac{d^2x_k}{dt^2} - \frac{dx_k}{dt}\frac{d^2x_i}{dt^2} \qquad (i, k = 2, 3, 4, 5). \tag{P}$$

To obtain the equations of the correspondence it is only necessary to find the ratio of the values of dx/dt and d^2x/dt^2. In order to do this we shall make use of the relation

$$\Sigma x_i = 1,$$

which renders the coordinates homogeneous.

Then to determine dx we have

$$\sum_i \frac{dx}{dt_i} = 0, \qquad \sum_i f_i \frac{dx}{dt_i} = 0,$$

$$\sum_{ik} y_i f_{ik} \frac{dx}{dt_k} = 0, \quad \sum_{jk} z_i f_{ik} \frac{dx}{dt_k} = 0,$$

from which we have

$$\frac{dx_1}{dt} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ f_2 & f_3 & f_4 & f_5 \\ \Sigma y_i f_{2i} & \cdot & \cdot \quad \cdot & \Sigma y_i f_{5i} \\ \Sigma z_i f_{2i} & \cdot & \cdot \quad \cdot & \Sigma z_i f_{5i} \end{vmatrix},$$

and expanding according to the minors of the last two rows,

$$\frac{dx_1}{dt} = \pm \sum_{i,j=1}^{i,j=4} \sum_{k,e=2}^{k,e=5} \begin{vmatrix} f_{ik} & f_{jk} \\ f_{il} & f_{jl} \end{vmatrix} (f_m - f_n) p_{ij},$$

where m, n are the remaining numbers of 2, 3, 4, 5 after k, l are chosen. The other derivatives can now be written down from symmetry

$$\frac{dx_h}{dt} = \pm \sum_{i,j=1}^{i,j=4} h \sum_{k,l=1}^{k,l=5} \begin{vmatrix} f_{ik} & f_{jk} \\ f_{il} & f_{jl} \end{vmatrix} (f_m - f_n) p_{ij},$$

where the letter h written in front of the second summation sign indicates that from the sequence 1, 2, 3, 4, 5 the number h is to be omitted. As above m, n are the remaining numbers of 1, 2, 3, 4, 5 after h, k, l are chosen. The expressions for d^2x/dt^2 can be obtained at once by differentiating. They are

$$\frac{d^2x_h}{dt^2} = \sum_{s=1}^{s=5} \sum_{i,j=1}^{i,j=5} h \sum_{k,l=1}^{k,l=5} \begin{vmatrix} f_{iks} & f_{jk} \\ f_{ils} & f_{jl} \end{vmatrix} (f_m - f_n) p_{ik} \frac{dx_s}{dt}$$

$$+ \sum_{s=1}^{s=5} \sum_{i,j=1}^{i,j=4} h \sum_{k,l=1}^{k,l=5} \begin{vmatrix} f_{ik} & f_{jks} \\ f_{il} & f_{jls} \end{vmatrix} (f_m - f_n) p_{ik} \frac{dx_s}{dt}$$

$$+ \sum_{s=1}^{s=5} \sum_{i,j=1}^{i,j=4} h \sum_{k,l=1}^{k,l=5} \begin{vmatrix} f_{ik} & f_{jk} \\ f_{il} & f_{jl} \end{vmatrix} (f_{ms} - f_{ns}) p_{ik} \frac{dx_s}{dt}.$$

Now making use of relations (13) we see that

$$\frac{dx_5}{dt} = 0,$$

since

$$\Sigma f_i \frac{dx_i}{dt} = 0.$$

Then the expressions for the derivatives become

$$\frac{dx_k}{dt} = \pm \sum_{i,j=1}^{i,j=4} h \sum_{k,l=1}^{k,l=5} \begin{vmatrix} f_{ik} & f_{jk} \\ f_{il} & f_{jl} \end{vmatrix} f_5 p_{ij},$$

and

$$\frac{d^2x_h}{dt^2} = \pm \sum_{s=1}^{s=5} \sum_{i,j=1}^{i,j=4} h \sum_{k,l=1}^{k,l=5} \left[\begin{vmatrix} f_{iks} & f_{jk} \\ f_{ils} & f_{jl} \end{vmatrix} f_5 + \begin{vmatrix} f_{ik} & f_{jks} \\ f_{il} & f_{jls} \end{vmatrix} f_5 + \begin{vmatrix} f_{ik} & f_{jk} \\ f_{il} & f_{jl} \end{vmatrix} (f_{ms} - f_{ns}) \right] p_{ij} \frac{dx_s}{dt}.$$

From these values the coordinates q_{ik} of the osculating planes can be calculated and we see at once that the correspondence is cubic.

5. Proceeding with the correspondence between the lines p_{ij} of π_3 and the planes q_{ik} passing through O, let us take O and three successive points of a curve passing through O and draw the four tangent spaces to V_3 at these points and put in the condition that the four points lie in the same plane q_{ik} and that the four tangent hyperplanes intersect in the same line p_{ij} of π_3. In this case the curve of contact of the cone circumscribed to V_3 with the line p_{ij} for vertex will have the plane q_{ik} for hyperosculating plane. Then the four points

$$x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \frac{d^3x}{dt^3},$$

will be connected by a linear relation

$$Ax + B\frac{dx}{dt} + C\frac{d^2x}{dt^2} + D\frac{d^3x}{dt^3} = 0,$$

which will require the vanishing of two four row determinants of the matrix

$$\left\| x \quad \frac{dx}{dt} \quad \frac{d^2x}{dt^2} \quad \frac{d^3x}{dt^3} \right\|$$

From the expression for d^2x/dt^2 it is at once seen that d^3x/dt^3 is of order three in the line coordinates p_{ij} and hence the determinants of the above matrix are of order six in p_{ij}. Therefore: *There is a congruence of lines of order* 36 *in* π_3 *such that any tangent cone having one of these lines for vertex will have a hyperosculating plane at the point of tangency of the* π_3.

By the same method of reasoning it is easily seen that in π_3 there are $6^2 \cdot 7^2$ lines such that the curves of contact of the tangent cones drawn to V_3 from these lines has in O a plane which has contact of order four, that is O and four successive points lie in one plane.

6. **Hypersurfaces in S_n.**—The preceding discussions can be applied to V_{n-1} in S_n without change. The lines passing through a point O of V_{n-1} which have three point contact with V_{n-1} form a quadric cone in the tangent S_{n-1}. This cone is the cone of united elements of the correspondence formed by taking the line joining two successive points of V_3 and making it correspond to the S_{n-2} of intersection of the tangent spaces to V_3 at those points. An S_{n-2} passing through O (and tangent to V_3) and its corresponding line are polar reciprocals with respect to the above cone. The osculating plane to the curves in the direction of the elements of the cone will lie in the tangent S_{n-1} at O. The directions along which the osculating S_3 of the curve lies in the tangent S_{n-1} form a cone of order 3 and dimensions $n - 3$. The elements have four-point contact with V_{n-1}. The directions along which the osculating S_4 to the curve lies in in the tangent S_{n-1} form a cone of order $\underline{|4}$ and dimensions $n - 4$ and the elements have five-point contact with V_{n-1}, etc. Finally there are $n - 1$ directions such that the osculating S_{n-1} to the curves will coincide with the tangent S_{n-1}. These directions have $(n - 1)$-point contact with V_{n-1} and form a generalization of asymptotic directions.

If we consider the curves on V_{n-1} which pass through O and take the tangent spaces S_{n-1} at three successive points and the plane which contains the three points then we establish a (1, 1) correspondence between the planes passing through O and the spaces S_{n-3} contained in the tangent S_{n-1} to V_3 at O. The correspondence is cubic as in S_4. There are ∞^{n-2} S_{n-3}'s such that the curve of intersection of the tangent cone drawn to V_3 from them has in O a hyperosculating plane. These S_{n-3}'s form an $(n - 2)$ parameter family of order 6^{n-2}. In the tangent S_{n-1} there are $6^{n-2} \cdot 7^{n-2}$ S_{n-3}'s such that the tangent cone to V_3 from one of them will be tangent along a curve which has in O a plane which has fourth order contact.

Massachusetts Institute of Technology,
November 14, 1910.